Multiaxial fatigue life prediction of rubber using configurational mechanics and critical plane approaches: a comparative study

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ABSTRACT: In the study of fatigue of elastomers, the crack nucleation approach is advantageous for analyzing the spatial distribution of fatigue life as it is based on quantities that are defined at a material point in the sense of continuum mechanics. Fatigue life is generally associated with the occurrence of a small crack on the specimen surface. Under the umbrella of this approach, two relatively newly-developed and promising methods are the configurational mechanics and the critical plane methods. While the configurational mechanics which captures material inhomogeneities, at once true and quasi material inhomogeneities, is entirely expressed in the material manifold, i.e. in the reference configuration, the critical plane method is expressed in the current configuration of the rubber body. In the present work, a comparative study on the performance of the two methods are addressed. Each method is assessed by considering experimental data available in literature and perspectives are drawn.

1 INTRODUCTION

In general, the fatigue life process involves a period during which cracks initiate in regions that were initially free of observed cracks, followed by a period during which nucleated cracks grow to the point of failure (Mars & Fatemi 2005). Two approaches currently available for predicting fatigue life in rubber are the crack growth (propagation) and the crack nucleation (initiation) approaches. In the crack growth approach, the behavior of a preexisting crack under mechanical loading is observed. The crack growth is determined by the calculation of the so-called tearing energy for given specimen and crack shapes and for prescribed loading conditions. This approach is thus suitable to the case where the crack path is well-identified (Saintier et al. 2006a). Another inconvenient of this approach is that the initial crack shape and position should be known, which is not possible in most engineering problems. Therefore, the crack nucleation approach seems more appropriate for evaluating fatigue life under complex loading and analyzing the spatial distribution of fatigue life in rubber body. Indeed, it offers simplicity and familiarity as it is based on quantities that are defined at each material point in the sense of continuum mechanics.

Under the umbrella of this approach, two relatively newly-developed and promising methods are the configurational mechanics and the critical plane approaches. This paper attempts to give a brief comparative study of efficiency of the two methods. A more comprehensive study will be provided in an upcoming paper.

2 GENERAL THEORY

2.1 Critical plane approach

The critical plane approach is based on physical observation that fatigue cracks initiate and grow within material on certain privileged planes called critical planes. The corresponding failure is supposed to be due to the stress, strain or energy histories acting on these planes. The critical plane is then determined by identifying material plane which maximizes the combination of relevant fatigue damage parameters expressed from the stress and/or strain histories. Following the pioneering work of Stanfield (1935), various multiaxial fatigue criteria can be found in the litera-
ture, see for example Findley (1959), Brown & Miller (1973) or Fatemi & Socie (1988) and the references herein.

While this approach enjoyed a great deal of success in metals, only few studies use the critical plane approach to predict fatigue failure in rubber materials. This is mainly due to the fact that multiaxial loading effects in rubber, which often undergoes large strain loading conditions, are not yet well understood (Mars & Fatemi 2002). Two works can be found in the literature which deal with the application of the critical plane approach to rubber fatigue: Saintier et al. (2006a) and Saintier et al. (2006b). In the following, these works are detailed.

In rubber materials under uniaxial fatigue loading condition, two opposite trends are observed. For negative loading ratio $R = \sigma_{\text{min}}/\sigma_{\text{max}}$, at fixed stress amplitude, fatigue life is observed to decrease as mean stress increases (Cadwell et al. 1940; Saintier et al. 2006b). This tendency is reversed as soon as $R$ becomes positive, i.e. fatigue life improves under non-relaxing tension cyclic loading. Even if the physical origin of this reinforcement is not well-known, it is often attributed to strain-induced crystallization, see for example the works of Cadwell et al. (1940), Gent (1994) or André et al. (1999). In order to incorporate these two mechanisms, i.e. damage and reinforcement, in the prediction of fatigue crack nucleation in rubber, Saintier et al. (2006b) proposed a Cauchy stress-based predictor, denoted $\Phi$, which is defined by:

$$\Phi = \frac{\Phi_{\text{damage}}}{1 + \Phi_{\text{reinforcement}}}$$  \hspace{1cm} (1)

where $\Phi_{\text{damage}}$ and $\Phi_{\text{reinforcement}}$ are two quantities associated with damage and reinforcement mechanisms respectively and to be computed from the Cauchy stress history. From uniaxial fatigue test results, it was shown that compressive stresses have a negligible effect on fatigue life. Thus, the author suggested that the maximum normal Cauchy stress acting on the critical plane $\sigma_{\text{n,cp}}$, defined by normal vector $\mathbf{n}_{\text{cp}}(t)$, controls the fatigue damage evolution. Thus, we have:

$$\Phi_{\text{damage}} = \sigma_{\text{n,cp}}^{\text{max}}$$  \hspace{1cm} (2)

and $\sigma_{\text{n,cp}}(t)$ is computed from:

$$\sigma_{\text{n,cp}}(t) = \mathbf{n}_{\text{cp}}(t) \cdot \mathbf{\sigma}(t) \mathbf{n}_{\text{cp}}(t)$$  \hspace{1cm} (3)

where $\mathbf{\sigma}$ is the Cauchy stress tensor. It is to note that due to large deformation effect, the orientation of the critical plane changes during loading. Therefore, the normal of the critical plane coincides with the eigenvector $\mathbf{v}(t)$ associated with the maximum principal Cauchy stress at time instant $t = t_{\text{max}}$, i.e.:

$$\mathbf{n}_{\text{cp}} = \mathbf{v}|_{t=t_{\text{max}}}$$  \hspace{1cm} (4)

where $t_{\text{max}}$ corresponds to the instant at which the maximum principal Cauchy stress over the period is reached. The unit normal vector of potentially occurring crack plane is determined by simply projecting the critical plane to the reference configuration:

$$\mathbf{n}_{\text{crack}} = \frac{\mathbf{F}^t \mathbf{n}_{\text{cp}}}{|\mathbf{F}^t \mathbf{n}_{\text{cp}}|}|_{t=t_{\text{max}}}$$  \hspace{1cm} (5)

Concerning reinforcement, Saintier et al. (2006b) postulated that the reinforcement effect is proportional to the minimum crystallinity level $X_c$ on the critical plane, i.e.:

$$\Phi_{\text{reinforcement}} = A \cdot X_c$$  \hspace{1cm} (6)

where $A$ is a proportionality constant. Based on the results of X-ray diffraction measurements, the authors suggested, in the fatigue reinforcement context, a relation between crystallization level and the applied stress as follow:

$$X_c(\sigma_{\text{reinf}}) = 0.3 \left[1 - \exp\left(-D(\sigma_{\text{reinf}} - \sigma_{\text{threshold}})\right)\right]$$  \hspace{1cm} (7)

where $\langle \cdot \rangle$ denotes the MacCauley bracket, $\sigma_{\text{reinf}}$ is an equivalent stress describing the reinforcement and $\sigma_{\text{threshold}}$ represents a threshold value for crystallization. To determine $\sigma_{\text{reinf}}$, it is proposed that reinforcement occurs as soon as the crack tip does not completely relax during a cycle. According to the authors, the latter takes place either when the minimum normal Cauchy stress over the period on the critical plane is positive, i.e. when $\sigma_{\text{n,cp}}^{\text{min}} = \min_t [\sigma_{\text{n,cp}}(t)] > 0$, or when the shear Cauchy stress $\tau_{\text{s,cp}}(t)$ on the critical plane has non-zero value at the crack closure, i.e. when $\tau_{\text{s,cp}}(t_{\text{cls}}) \neq 0$ for $\sigma_{\text{n,cp}}(t_{\text{cls}}) = 0$. Suppose that $\sigma_{\text{n,cp}}(t)$ reaches its minimum value at $t_{\text{min}}$, i.e. $\sigma_{\text{n,cp}}^{\text{min}} = \sigma_{\text{n,cp}}(t_{\text{min}})$, thus (see Saintier et al. (2006b) for details):

$$\sigma_{\text{reinf}} = \begin{cases} \sqrt{[\sigma_{\text{n,cp}}(t_{\text{min}})]^2 + [\tau_{\text{s,cp}}(t_{\text{cls}})]^2} & \text{if } \sigma_{\text{n,cp}}^{\text{min}} > 0, \\ \tau_{\text{s,cp}}(t_{\text{cls}}) & \text{otherwise.} \end{cases}$$  \hspace{1cm} (8)

and $\tau_{\text{s,cp}}(t)$ is simply given by:

$$\tau_{\text{s,cp}}(t) = \sqrt{\|\mathbf{\sigma}(t)\mathbf{n}_{\text{cp}}(t)\|^2 - (\sigma_{\text{n,cp}}(t))^2}$$  \hspace{1cm} (9)

Finally, the number of cycles to crack initiation $N_f$ is correlated to $\Phi$ via a classical power law function:

$$N_f = \left(\frac{\Phi}{\Phi_0}\right)^\alpha$$  \hspace{1cm} (10)

where $\Phi_0$ and $\alpha$ are material parameters identified from uniaxial tension fatigue data.

In the following, this predictor, given in Eqs. (1) to (9), will be referred to as the P-01 predictor.
2.2 Configurational mechanics

To some extent, the presence of defects, e.g. crack, inclusion, cavities..., within materials can no longer be neglected in the modeling of their mechanical responses under various loading conditions. In the case of macroscopic crack, the concept of fracture mechanics, following pioneering works of Inglis (1913) and Griffith (1920), offers an efficient theoretical foundation. The crack growth is determined by the calculation of the energy release rate for given specimen and crack shapes and for prescribed loading conditions. The major difficulty of this approach is that initial crack shape and position should be known, which is not possible in most engineering problems. Moreover, when dealing with materials containing uniformly distributed microscopic cracks or microscopic defects in general, the notion of the energy release rate is then not straightforward. Thus, a continuum approach which accounts for existence of material inhomogeneities is needed.

In fact, a general and efficient way to analyze different kinds of material inhomogeneities within the framework of continuum mechanics is provided by the theory of configurational mechanics (Gross et al. 2003; Maugin 1995). The introduction of the Configurational Mechanics, also designated as the Eshelbian Mechanics by Maugin (1993) and the Mechanics in Material Space by Kienzler & Herrmann (2000) by contrast to the Newtonian Mechanics or the Mechanics in Physical Space respectively, date back to the outstanding work of Eshelby (1951). While in classical Newtonian Mechanics, attention is focused on physical forces generated by displacements in physical space, i.e. the three-dimensional Euclidean space \(E^3\), in Eshelbian Mechanics, we deal with a different class of forces, referred to as configurational forces, which are generated by displacements not in the physical space but in the material space (or manifold) \(M^3\), i.e. the abstract set of particles that constitute the body (Truesdell & Noll 1965).

Generally, material motion in the physical space induces microstructural changes or rearrangements in the material, e.g. growth of microscopic defects, dislocation or displacement of boundary phases. Describing such rearrangement in the physical space is not an easy task (Steinmann 2000). Thus, the balance of physical linear momentum (in Lagrangian or Eulerian description) has to be completely written onto the material space (Maugin 1995). The corresponding equation is given by:

\[
\text{Div}_X \Sigma + G = 0
\]  

and

\[
G = - \frac{\partial W}{\partial X_{\text{expl}}}
\]

In Eq. (12), \(\Sigma\) is the configurational stress tensor (or the energy momentum tensor or the Eshelby stress tensor) and \(I\) is the \(3 \times 3\) identity tensor. In Eq. (12), \(S\) is the second Piola-Kirchhoff stress tensor and \(C\) is the right Cauchy-Green strain tensor equal to \(F^T F\). Moreover, in Eq. (13) \(G\) is referred to as the configurational force vector associated with inhomogeneities. It is defined as the negative explicit differentiation of the strain energy with respect to the particle position in the material manifold (see the index \(\cdot_{\text{expl}}\)). If the material is assumed homogeneous, the configurational force \(G\) vanishes and the configurational stress tensor satisfies a strict conservation law:

\[
\text{Div}_X \Sigma = 0
\]

In the majority of studies involving Configurational Mechanics, only configurational forces are investigated through the calculation of path-independent integrals around inhomogeneities. So, configurational stress only appears in the definition of surface tractions, i.e. after contraction with the outward normal of the contour. Most of these works focus on the application of Configurational Mechanics to Fracture Mechanics (see brief literature survey of Steinmann (2000) or Verron & Andriyana (2007) and the references herein).

Opposite to the case of configurational forces, only few studies are concerned with the peculiar properties of the configurational stress tensor \(\Sigma\). Considering the geometrical definition and the physical significance of the Cartesian components of this tensor, it appears that the configurational stress is a continuum mechanics quantity associated with energy changes during local structural rearrangement of material under loading (Epstein & Maugin 1990; Kienzler & Herrmann 1997). As fatigue loading conditions induce significant microstructural rearrangements in rubber, Verron & Andriyana (2007) and Andriyana & Verron (2007) consider this tensor as an appropriate continuum mechanics quantity to derive a new predictor for rubber fatigue. By supposing that opening and closing of microscopic defects (cavities) in rubber are due to only material normal traction and not due to material shear, the authors proposed that microscopic defects growth can be predicted by considering the smallest eigenvalue of this tensor. Thus the predictor is given by:

\[
\Sigma^* = \left| \min \left( \Sigma_i \right)_{i=1,2,3} , 0 \right| \geq 0
\]

where \(\Sigma_i\) are the principal configurational stresses. When one (or more) principal stress is negative, the predictor is strictly positive, and the defect
tends to grow and to turn into a plane crack orthogonal to $\mathbf{V}^*$, the eigenvector associated with $-\Sigma^*$. When the three principal stresses are positive, the material tractions tend to shrink the flaw and the predictor is set to 0.

In order to incorporate non-proportional multiaxial loading conditions, the authors proposed to accumulate the increment of the configurational stress that contributes to flaw opening. In this case, the previous predictor becomes (see Verron & Andriyana (2007) for details):

$$\Sigma^* = \min \left[ \left( \Sigma^d \right)_{i=1,2,3} : 0 \right]$$  \hspace{1cm} (16)

where $(\Sigma^d)_{i=1,2,3}$ are the eigenvalues of the damage part of the configurational stress tensor $\Sigma^d$. This tensor is obtained by the integration over the cycle of:

$$d\Sigma^d = \sum_{i=1}^{3} d\Sigma^d_i \mathbf{V}_i \otimes \mathbf{V}_i$$  \hspace{1cm} (17)

with:

$$d\Sigma^d_i = \begin{cases} \frac{d\Sigma_i}{dC} & \text{if } d\Sigma_i < 0 \text{ and } \mathbf{V}_i \cdot \Sigma \mathbf{V}_i < 0, \\ 0 & \text{otherwise,} \end{cases}$$  \hspace{1cm} (18)

$(d\Sigma_i)_{i=1,2,3}$ and $(\mathbf{V}_i)$ being the eigenvalues and eigenvectors of the configurational stress tensor increment:

$$d\Sigma = \frac{d\Sigma}{dC} : dC$$  \hspace{1cm} (19)

It is to note that for fully-relaxing proportional loading conditions, the integration over one cycle reduces to the determination of the instantaneous value of the configurational stress tensor for the maximum strain level. In this case, Eq. (16) reduces to Eq. (15).

Finally, the number of cycles to crack initiation $N_f$ is correlated to $\Sigma^*$ via a classical power law function:

$$N_f = \left( \frac{\Sigma^*}{\Sigma_o} \right)^{\beta}$$  \hspace{1cm} (20)

where $\Sigma_o$ and $\beta$ are material parameters identified from uniaxial tension fatigue data.

In the following, this predictor, given in Eqs. (16) to (19), will be referred to as the P-02 predictor.

3 RESULTS AND DISCUSSION

In order to assess the efficiency of the two predictors, the experimental data of Mars (2001) are considered. The author conducted number of axial tensile/torsion fatigue experiments under proportional and non-proportional loading conditions. To calculate each predictor, the analytical solution of the simultaneous axial tensile and torsion of a hyperelastic cylinder is adopted (see for example in Green & Adkins (1960)). As proposed by Mars, the material is assumed to follow the Neo-Hookean model:

$$W = C \left( I_1 - 3 \right)$$  \hspace{1cm} (21)

with $C = 1.5$ MPa. It is to note that for the P-01 predictor, non-proportional loading conditions allow to account for the reinforcement mechanism, i.e. $\Phi_{\text{reinforcement}} \neq 0$. Contrary, this mechanism is not incorporated in the P-02 predictor. Indeed, it was shown that the reinforcement can be taken into account when the rubber behavior is modeled using a simple visco-hyperelastic constitutive equation in order to capture hysteretic response (Andriyana & Verron 2007). For non-relaxing loading conditions, the P-02 predictor is computed using Eq. (16). Results will be presented in the form of the Wöhler curve and fatigue life prediction.

In Figures 1 and 2, the Wöhler curves are plotted using the P-01 and P-02 predictors respectively. In order to compute theoretical fatigue life, material constants in Eqs. (10) and (20) are fitted from simple tension loading condition, i.e. path A. The corresponding fatigue life predictions are given in Figures 3 and 4 for P-01 and P-02 predictors respectively.

Figure 1: Wöhler curve using the P-01 predictor.

In general the two approaches give relatively good prediction. Different fatigue data (paths A, C, D, E, H, I) in Figures 1 and 2 are well-unified even if the P-02 predictor appears to give a slightly better results. The superiority of the P-01 over the P-02 is shown...
in the path I results (non-proportional tension/torsion with 180° phase angle) of Figure 3, while the superiority of the P-02 over the P-01 is highlighted in the paths B (simple torsion) and F (proportional compression/torsion) of Figure 4 which correspond to cases where the torsion loading is predominant.

4 CONCLUSIONS

This paper attempted to give a brief comparative study of efficiency of two approaches used in predicting rubber fatigue: critical plane and configurational mechanics. General theory of each approach was briefly presented. As a first comparison, the experimental data of Mars (2001) were considered. First results show that both methods give relatively good fatigue prediction. Slightly better results are obtained using the P-02 predictor for torsion loading. Nevertheless, more experimental data are needed to give a better idea of the efficiency of each predictor.

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